Self-organized critical scaling at surfaces

A. L. Stella,[1] C. Tebaldi,[1] and G. Caldarelli[2]
[1]Dipartimento di Fisica e Sezione Istituto Nazionale di Fisica Nucleare, Università di Padova, Via F. Marzolo 8, I-35131 Padova, Italy
[2]Scuola Internazionale Superiore di Studi Avanzati–International School for Advanced Studies, Via Beirut 2, I-34100 Miramare, Trieste, Italy
(Received 26 August 1994)

At dissipative boundaries, models of self-organized criticality show peculiar scalings, different from the bulk ones, in the distributions characterizing avalanches. For Abelian models with Dirichlet boundary conditions, evidence of this is obtained by a mean field approach to semi-infinite sandpiles, and by numerical simulations in two and three dimensions. On the other hand, within the mean field description, closed Neumann conditions restore bulk scaling exponents also at the border. Numerical results are consistent with this property also at finite d.

PACS number(s): 05.40.+j, 05.60.+w, 68.35.Rh, 64.60.Ht

Bak, Tang, and Wiesenfeld (BTW) [1] introduced the concept of self-organized criticality (SOC) as an explanation of the widespread occurrence in nature of power law correlations in space and time. Relatively simple automaton models, inspired by the physics of real sandpiles [2], were shown by BTW to possess the remarkable property of evolving stochastically into a critical, stationary state, independent of initial conditions. Such sandpile models can be applied to very diverse phenomena, ranging from earthquakes [3] to magnetic domain patterns [4].

An essential requirement of any sandpile model is the capability of dissipation at the borders, which allows the system to reach the critical state by elimination of the "sand" injected at random in the system. So far, in spite of the clear importance played by boundary conditions in SOC, very little attention has been paid to the possible consequences of the borders on critical behavior. In the context of standard critical phenomena, inhomogeneities, of which a boundary surface is a typical example, are well known to produce exponents different from those describing bulk singularities [5]. The boundary magnetization of a semi-infinite spin system, for example, responds to the bulk field with a susceptibility $\chi_1 \sim |(T-T_c)/T_c|^{-\gamma_1}$ for $T \to T_c$, with γ_1 different from the bulk susceptibility exponent γ [5]. This implies that the surface magnetic field, on which the boundary free energy depends, has a peculiar scaling dimension, different from that of the bulk field.

In the present paper we provide evidence that one can clearly distinguish between bulk and surface scaling exponents in SOC systems, in the same way as in equilibrium spin models. In spite of the considerable recent activity on SOC, the general possibility of peculiar scalings at the boundary has not been appreciated. In a recent study [6] of a particular one-dimensional (1D) SOC model, evidence was produced of different bulk and edge exponents, as well as multiscaling behavior, for a "through distribution." Most recently, boundary height-height correlations were computed for the stationary state of an Abelian sandpile model (ASM) [7]. Based on conformal invariance it was shown that the exponents are the same as for the corresponding bulk correlations. This identity can be attributed to the particular (energy) zero-state Potts operator involved [7]. Indeed, an exact correspondence exists between the statistics of this last model and the stationary critical state [8]. However, such a mapping does not allow discussion of the avalanche distribution exponents, which determine how this state responds to the external flux. The values of these last exponents remain an open theoretical challenge. As we show below, it is indeed in avalanche distributions that novel boundary scaling behavior is revealed.

We first consider the ASM, for which much analytical and numerical information is already available [8–10]. At each site i of a finite box Λ in the d-dimensional hypercubic lattice, an integer variable $z_i = 0,1,2,3,\ldots,$ is defined. If $z_i \le 2d-1$ for all i, the configuration is stable. As soon as some $z_k \ge 2d$, a toppling occurs, so that $z_i \to z_i - \Delta_{ki}$, where Δ is the Laplacian matrix: $\Delta_{ll} = 2d$ and $\Delta_{lm} = -1$ if $l \ne m$ and l,m are nearest neighbors, and 0, otherwise. For $j \in \partial\Lambda$, the boundary of Λ, the standard condition, allowing sand grain elimination, is of the Dirichlet form: $\Delta_{jj} = 2d$ for $j \in \partial\Lambda$. Of course, nothing prevents us from considering closed, Neumann boundary conditions (BC's), i.e., $\Delta_{jj} = 2d-1$ for j belonging to a subset of $\partial\Lambda$, e.g., one side or face. For such j's the stability condition then becomes $z_j \le 2d-2$. Closed BC's alone do not allow sand elimination, which takes place through the other Dirichlet faces. The dynamics is as follows. Starting from a stable configuration, a site k is selected at random and a grain is added: $z_k \to z_k + 1$. If the configuration remains stable, no avalanche is produced, and a new grain is injected. If the site becomes unstable, it topples. Other sites might become unstable as a consequence of this first toppling, and, in a second stage, one lets all such sites topple. This proceeds stage by stage until a stable configuration is eventually reached. During the avalanche, time may be measured in units of the interval between successive checks of unstable points.

Avalanches are thus naturally characterized by the duration t, the number of sites topping at least once, s, and the number of topplings, m. For such quantities, scaling distributions are expected under stationary conditions [1,8]. So, if $L \gg 1$ is the side length of a cubic Λ, consistently with finite size scaling (FSS), one finds $P(s,L) \simeq L^{-\tau\bar{d}} g(s/L^{\bar{d}})$ and $P(s,\infty) \simeq s^{-\tau}$ for the probability density of avalanches involving s sites. $\bar{d}$ is the fractal dimension of the avalanches. Analogously the t and m distributions scale as t^{-y} and $m^{-\tau_m}$, respectively, in the infinite system. One can also define a susceptibility [11]

$$\chi(L) \equiv \int_1^\infty P(s,L) s \, ds \sim L^{\bar{d}(2-\tau)} . \tag{1}$$

If θ denotes the average height $\langle z \rangle$ of the infinite pile out of criticality, Eq. (1) and FSS lead to

$$\chi(\theta) \sim R^{\bar{d}(2-\tau)} \sim |\theta - \theta_c|^{-\nu\bar{d}(2-\tau)} , \tag{2}$$

where $R \sim |\theta-\theta_c|^{-\nu}$ is the average linear size of avalanches and θ_c is the average height at criticality [11]. Thus the susceptibility exponent is $\gamma = \nu\bar{d}(2-\tau)$.

So far, the determinations of τ, ν, and $\bar{d}$ in the literature were all based on sampling avalanches starting anywhere in the whole system and thus essentially in the bulk, but, as shown below, including surface contributions. We propose to define the analog of a boundary magnetic susceptibility by sampling only avalanches that start at the border $\partial\Lambda$. In analogy with standard critical phenomena, we expect for these avalanches a $P_{\rm sur}(s,L)$, obeying FSS with $\tau_{\rm sur}$ and $g_{\rm sur}$ possibly different from τ and g, respectively, and the same $\bar{d}$.

One can gain analytical insight into this issue through a mean field (MF) approach to the ASM in semi-infinite geometry, which extends and generalizes a recently proposed "reaction rate" method for bulk quantities [12]. The MF approach allows us to treat inhomogeneity in the simplest 1D setting. The integer index $i = 1,2,\ldots,$ numbers sites starting from the origin of a semi-infinite chain, and P_{ki} ($k = 0,1,2,3,\ldots,$) represents the probability that $z_i = k$. The probability that $z_l \to z_l + 1$ as consequence of grain injection from the exterior is denoted by h. Reaction rate equations can then be written expressing the stationary state condition, i.e., the fact that, at each site, the rate of transitions out of a given height state exactly balances the rate into the same state. Since toppling occurs only for $z \geq 2$, it turns out that the critical stationary state is the one in which $P_{0n} = P_{1n} = \frac{1}{2}$, $P_{in} = 0$ for $i \geq 2$ and all n. In this state $\langle z \rangle = \theta_c = \frac{1}{2}$. P_{2n} is in fact the local order parameter. To leading order in P_{2n} one can put $P_{kn} \simeq \frac{1}{2} + \alpha_{kn} P_{2n}$, $k = 0,1$, where α's are suitable coefficients and $P_{3n} = O(P_{2n}^2)$. So, the following equations can be obtained eventually:

$$\begin{aligned} P_n = & -\tfrac{1}{2} P_{n-1} P_{n+1} - \frac{h}{2}(P_{n-1} + P_{n+1}) \\ & + (\theta - 2P_n)(P_{n-1} + P_{n+1}) \\ & + \tfrac{1}{4} P_n (P_{n-2} + P_{n+2}) + h(\theta - 2P_n) \\ & + \frac{h}{4}(P_{n-2} + 2P_n + P_{n+2}), \quad n \geq 2 \end{aligned} \tag{3a}$$

$$P_1 = \tfrac{1}{2} P_2 + (\theta - \tfrac{1}{2} - 2P_1) P_2 + h(\theta - 2P_1) - \frac{h}{2} P_2 , \tag{3b}$$

where for simplicity P_j stays for P_{2j} everywhere, and the α's have been eliminated by use of $\alpha_{0n} + \alpha_{1n} + 1 \simeq 0$ (probability conservation), and $\theta \simeq \frac{1}{2} + \alpha_{1n} P_{2n} + 2P_{2n}$. Equations (3a)–(3b) are valid in the case of Dirichlet BC's. By disregarding Eq. (3b) and turning to our standard notations, a translationally invariant solution of Eq. (3a), with $P_{2n} \equiv P_2$, satisfies

$$4P_2^2 + (1 - 2\theta + 2h) P_2 - h\theta = 0 , \tag{4}$$

which is easily seen to imply $P_2 \sim (\frac{1}{2})(\theta - \theta_c)$, $\theta \geq \theta_c$, i.e., $\beta = 1$ and $\chi \equiv \partial P_2 / \partial h \sim (\frac{1}{4})(\theta - \theta_c)^{-1}$, i.e., $\gamma = 1$ [12]. Inhomogeneous solutions of Eqs. (3a) and (3b) are most easily discussed in terms of $\chi_n \equiv \partial P_{2n} / \partial h|_{h=0}$. One finds a solution of the form

$$\chi_n = \chi + \delta \exp[-(n-1)q] , \tag{5}$$

with $q \sim |\theta - \theta_c|^{1/2}$ and $\chi_1 \sim |\theta - \theta_c|^{-1/2}$. Thus $\gamma_1 = \frac{1}{2} \neq \gamma$ in the MF, while the behavior of the reciprocal "penetration length" q reveals directly the expected $\nu = \frac{1}{2}$ [12]. Since $\bar{d} = 4$ in the MF [12], we conclude that $\tau_{\rm sur} = 2 - \gamma_1 / \nu\bar{d} = \frac{7}{4}$, to be compared with the bulk $\tau = \frac{3}{2}$. A similar calculation gives γ_1 with the Neumann BC at site 1. In this case P_{11} replaces P_{21} everywhere and Eq. (3b) is suitably modified. Again a solution of the form (5) is found, this time with $\chi_1 \equiv \partial P_{11} / \partial h|_{h=0} \sim |\theta - \theta_c|^{-1}$. A subleading term proportional to $|\theta - \theta_c|^{-1/2}$ is also obtained in χ_1. Thus, in the MF closed conditions give $\gamma_1 = \gamma = 1$. MF calculations can of course be carried on also in higher d, yielding the same γ_1's. A more complete and detailed account of these MF results will be given elsewhere.

We performed extensive numerical investigations of s and t distributions for avalanches starting at both Dirichlet and Neumann boundaries, in 2D ($16 \leq L \leq 256$) and 3D ($16 \leq L \leq 40$). In the simulations the piles were kept critical by injecting grains over the whole system, thus making it unusually difficult to obtain good statistics on boundary events. Figure 1 shows data for $P_{\rm sur}$ and P with the Dirichlet BC in 2D. In this and the other cases also, time distribution data and radii of gyration of avalanches were recorded. Direct and data collapse fits allow us to estimate consistently $\tau_{\rm sur} = 1.52 \pm 0.05$ and $\bar{d} = 1.97 \pm 0.06$, to be compared with $\tau = 1.21 \pm 0.02$ and $\bar{d} = 1.96 \pm 0.06$, as obtained from our analysis in the bulk. Our bulk results agree rather well with determinations in the literature [10]. The time distribution exponent of border avalanches can be estimated as $y_{\rm sur} = 1.81 \pm 0.04$, to be compared with the bulk $y = 1.32 \pm 0.04$.

On the basis of scaling arguments normally applied to the bulk, we expect $z_{\rm sur} = \bar{d}(\tau_{\rm sur} - 1)/(y_{\rm sur} - 1) = 1.26 \pm 0.08$, where $z_{\rm sur}$ is the exponent relating the duration of border avalanches with their linear range. Our determination of $z_{\rm sur}$ is consistent with the estimated ($z/\bar{d} = 0.607 \pm 0.040$ [10]) and conjectured ($z = \frac{5}{4}$ [8]) bulk values. This consistency can be understood. Indeed, in 2D the bulk z is expected to coincide with the fractal dimension of red bonds [13] in the zero-state Potts spanning clusters [8]. Such a dimension should remain unaltered whether or not the cluster backbone is supposed to connect points on the boundary [14]. For $d=3$ we find $\tau_{sur}=1.73\pm0.05$, $\bar{d}=2.9\pm0.1$, and $y_{sur}=2.24\pm0.08$. In the bulk we find $\tau=1.40\pm0.05$, $y=1.61\pm0.04$, and $\bar{d}=2.9\pm0.1$. τ_{sur} is thus closer than τ to its MF value.

Avalanches starting on the border of the ASM with the Dirichlet BC give extra simplifications compared to the bulk ones. One can indeed argue that, if toppling first starts on $\partial\Lambda$, no sites of the avalanche will undergo multiple topplings [15]. Thus, in this case there is no distinction between s and m distributions. This corroborates the expectation that in general $\tau=\tau_m$, in the bulk too, as suggested by numerical results [16].

Within our accuracy, we verified that, at a face or side satisfying Neumann conditions, τ_{sur} could even coincide with the bulk τ. For example, an accurate determination in 2D gave $\tau_{sur}=1.27\pm0.05$ and $y_{sur}=1.41\pm0.05$ with this BC. This result suggests that, for ASM models, the two types of BC considered do not determine universal exponents, and the MF result $\gamma_1=\gamma$, $\tau_{sur}=\tau$ for the Neumann BC could hold also in finite d. This last possibility is rather intriguing, because it would imply that the presence of the closed border does not alter avalanche scaling. Of course, further and more extensive numerical work will be needed to possibly corroborate this conjecture.

As an example of a non-Abelian sandpile on which to base the study of boundary critical scaling, we chose the restricted critical Laplacian model (CLM), which has been clearly established to belong to a universality class different from that of the ASM [10]. Also for this model, where toppling at site i requires that both z_i and the Laplacian $(\nabla^2 z)_i=\sum_j^{2d} z_j-2dz_i$ simultaneously exceed thresholds, in 2D we found clear evidence of surface scaling distinct from bulk scaling. We applied the Dirichlet BC and the convention that the Laplacian at boundary

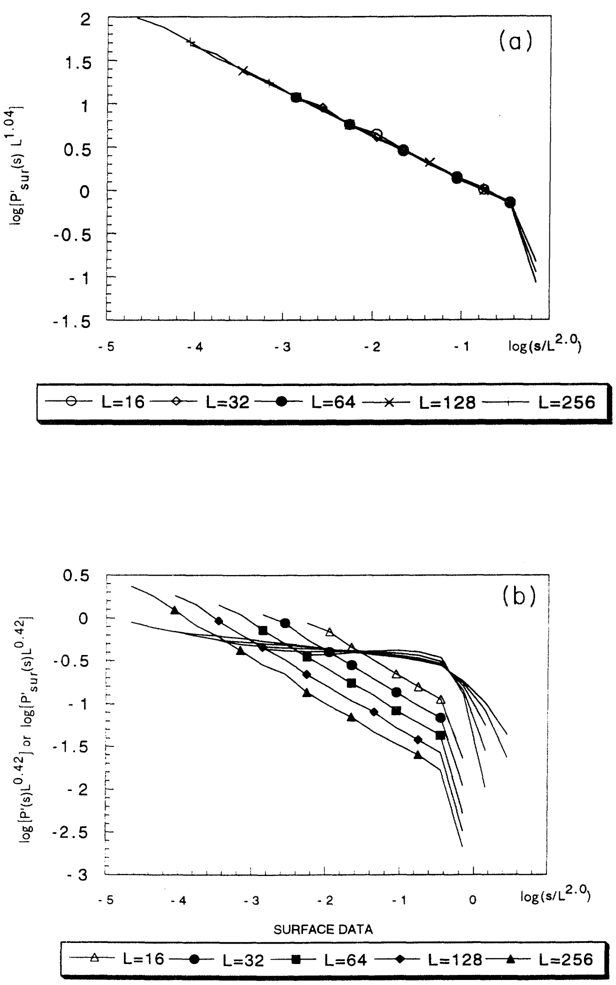

FIG. 1. (a) Data collapse for $P'_{sur}(s,L)=\int_1^s P_{sur}(s',L)ds'$ of the 2D ASM. P'_{sur} is scaled by $L^{\bar{d}(\tau-1)}=L^{1.04}$, corresponding to $\tau_{sur}=1.52$ and $\bar{d}=2$. For convenience not all data points are drawn. A direct estimate of $\bar{d}$ based on the log-log fitting of the radius of gyration data versus s gives $\bar{d}\simeq1.97\pm0.06$. (b) Plots for P'_{sur} and $P'(s,L)=\int_1^s P(s',L)ds'$, with rescaling corresponding to bulk τ and $\bar{d}$ values; surface data clearly do not collapse in this case. Bulk data points are not drawn explicitly. Logarithms are natural.

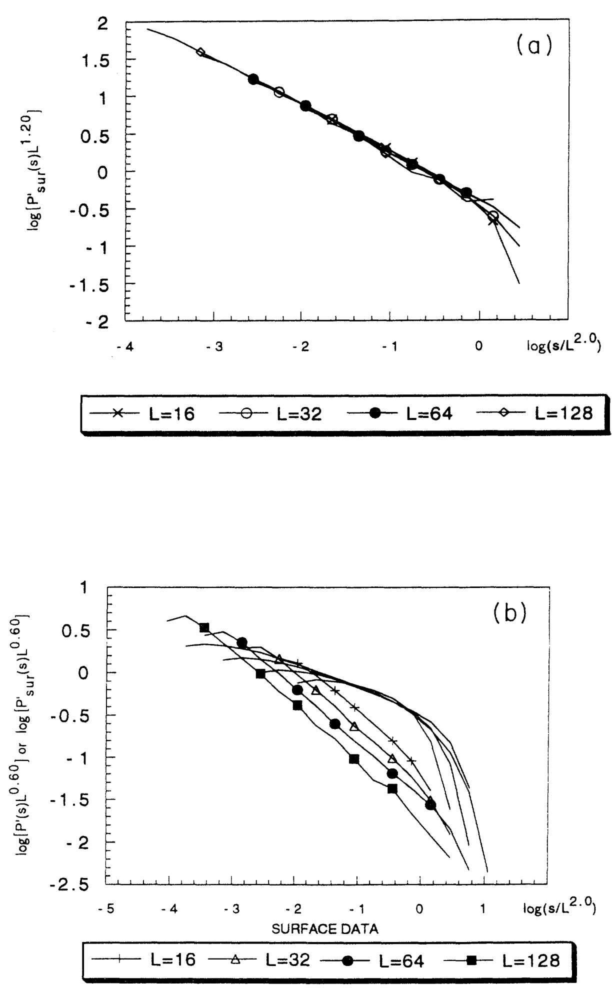

FIG. 2. (a) and (b) Same as in Figs. 1(a) and 1(b) for the 2D restricted CLM.

sites is computed by assuming a fictitious nearest neighbor site with $z=0$ outside the system. As illustrated also in Fig. 2, we got $\tau_{sur}=1.62\pm0.06$, $y_{sur}=1.97\pm0.03$, and $\bar{d}=1.93\pm0.05$, to be compared with $\tau=1.31\pm0.04$, $y=1.48\pm0.03$, and $\bar{d}=1.94\pm0.04$ in the bulk. Also in this case, when replacing the Dirichlet with the Neumann BC, the values of surface exponents become different and closer to the bulk ones. Indeed, we got in this case $\tau_{sur}=1.40\pm0.06$ and $y_{sur}=1.55\pm0.05$.

Summarizing, we have provided analytical and numerical evidence that SOC models, with dissipative BC, display peculiar surface scaling in their avalanche distributions. This shows that the SOC state mimics the structure of the ordinary critical state in a more complete way than has generally been realized. The existence of boundary scaling, which amounts to a sort of scaling correction for bulk behavior, should be taken into account, especially when analyzing results for relatively small samples. It should also be remembered that, for SOC, the fully periodic BC often used in spin problems to get rid of boundary effects would not make sense. The same applies to the Neumann BC's which, especially in the Abelian case, were found here to induce surface exponents very close, if not identical, to the bulk ones. Similar considerations can be made concerning the phenomenological analysis of SOC, e.g., in earthquakes. For example, taking into account the possible presence of geologically relevant boundaries in the selection of seismic events could prove important for the discussion of Gutenberg-Richter law exponents [3].

The existence of a well defined surface scaling in SOC also provides an unexpected expansion of the field in which theory and experiments should be compared. Thinking of the recent advances in the context of 2D equilibrium statistical models [17], for example, we can only foresee accelerated progress and a deeper understanding from such an expansion. By simultaneously focusing on both bulk and surface scalings, many universality issues have a better chance to be settled, at least at the numerical level.

Note added. After we submitted the present article, a paper appeared [E. V. Ivashkevich, D. V. Ktitarev, and V. B. Priezzhev, J. Phys. A **27**, L585 (1994)], in which $\tau_{sur}=\frac{3}{2}$ is predicted exactly for the ASM with the Dirichlet BC on a 2D lattice. The derivation of Ivashkevich *et al.*, besides using very plausible scaling assumptions, is based on an extension of the mapping of recurrent ASM configurations onto spanning trees [8]. Another important ingredient is the fact, also shown here, that in a boundary avalanche with the Dirichlet BC every site topples at most once. Our numerical estimate is pretty consistent with $\tau_{sur}=\frac{3}{2}$.

This work was partly supported by INFM, Unita' di Padova. M. Kardar, T.L. Einstein, and N.C. Bartelt are acknowledged for useful discussions.

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[14] At least in the case of the $q=1$ Potts model, i.e., percolation, this can be easily verified by considering the derivation [A. Coniglio, J. Phys. A **15**, 3829 (1982)] of the red bond dimension.

[15] Sites not belonging to $\partial\Lambda$ cannot be the first to topple twice. Indeed, for them a second toppling would require that at least one of their 2D neighbors have already toppled at least twice. On the other hand, also a site on $\partial\Lambda$ cannot be the first one to topple twice, since it has only 2D-1 neighbors that have toppled at most once before.

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